

GALOIS DESCENT OF ADDITIVE INVARIANTS

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ABSTRACT. Making use of the recent theory of noncommutative motives, we prove that every additive invariant satisfies Galois descent. Examples include (nonconnective) algebraic K -theory, cyclic homology (and all its variants), topological cyclic homology, *etc.*

1. INTRODUCTION

Additive invariants. A *differential graded (=dg) category* \mathcal{A} , over a base field k , is a category enriched over complexes of k -vector spaces; see §2. Let us denote by $\mathbf{dgc}at$ the category of (small) dg categories. Every (dg) k -algebra A gives naturally rise to a dg category \underline{A} with a single object and (dg) k -algebra of endomorphisms A . Another source of examples is provided by k -schemes since, as proved by Lunts-Orlov [8], the derived category of perfect complexes $\mathcal{D}_{\text{perf}}(V)$ of every (quasi-compact separated) k -scheme V admits a unique dg enhancement $\mathcal{D}_{\text{perf}}^{\text{dg}}(V)$.

Given a dg category \mathcal{A} , let us denote by $T(\mathcal{A})$ the dg category of pairs (i, x) , where $i \in \{1, 2\}$ and x is an object of \mathcal{A} . The complex of morphisms in $T(\mathcal{A})$ from (i, x) to (i', x') is given by $\mathcal{A}(x, x')$ if $i \geq i'$ and is zero otherwise. Composition is induced by the composition operation in \mathcal{A} ; consult [14, §4] for further details. Intuitively speaking, $T(\mathcal{A})$ “dg categorifies” the notion of upper triangular matrix. Note that we have two inclusion dg functors $i_1 : \mathcal{A} \hookrightarrow T(\mathcal{A})$ and $i_2 : \mathcal{A} \hookrightarrow T(\mathcal{A})$.

Definition 1.1. Let $E : \mathbf{dgc}at \rightarrow \mathbf{D}$ be a functor with values in an additive category. We say that E is an *additive invariant* if it satisfies the following two conditions:

- (i) it sends *Morita equivalences* (see §2) to isomorphisms;
- (ii) given any dg category \mathcal{A} , the inclusion dg functors induce an isomorphism¹

$$[E(i_1) \ E(i_2)] : E(\mathcal{A}) \oplus E(\mathcal{A}) \xrightarrow{\sim} E(T(\mathcal{A})).$$

Thanks to the work of Blumberg and Mandell, Keller, Schlichting, Thomason and Trobaugh, Waldhausen, Weibel, and others (see [1, 4, 5, 6, 10, 12, 13, 17, 18, 19]), examples of additive invariants include connective algebraic K -theory (K), nonconnective algebraic K -theory (\mathbb{K}), homotopy algebraic K -theory (KH), Hochschild homology (HH), cyclic homology (HC), periodic cyclic homology (HP), negative cyclic homology (HN), topological Hochschild homology (THH), and topological cyclic homology (TC). Recall from *loc. cit.* that when applied to \underline{A} , respectively to $\mathcal{D}_{\text{perf}}^{\text{dg}}(V)$, these invariants reduce to the classical invariants of (dg) k -algebras, respectively of k -schemes.

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¹Condition (ii) can be equivalently formulated in terms of semi-orthogonal decompositions in the sense of Bondal-Orlov; see [14, Thm. 6.3(4)].

Galois descent. Let l/k be a finite Galois field extension of degree $n := [l : k]$ with Galois group $G := \text{Gal}(l/k)$. Given a dg category \mathcal{A} , let us denote by \mathcal{A}_l the dg category obtained from \mathcal{A} by tensoring each complex of morphisms with l . Note that \mathcal{A}_l comes equipped with an action of G and with a canonical dg functor $\iota_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}_l$. Our main result is the following:

Theorem 1.2. (*Galois descent*) *Let $E : \text{dgcat} \rightarrow \mathbf{D}$ be an additive invariant with values in an idempotent complete $\mathbb{Z}[1/n]$ -linear additive category. Then, for every dg category \mathcal{A} one has a canonical isomorphism*

$$E(\iota_{\mathcal{A}}) : E(\mathcal{A}) \xrightarrow{\sim} E(\mathcal{A}_l)^G.$$

Intuitively speaking, Theorem 1.2 shows us that every additive invariant satisfies Galois descent as long as one inverts the degree of the field extension. The proof is based on the recent theory of noncommutative motives; consult §3–4 for details. By applying Theorem 1.2 to the above examples of additive invariants we obtain the following (concrete) isomorphisms:

Corollary 1.3. *For every (dg) k -algebra A and for every (quasi-compact separated) k -scheme V one has canonical isomorphisms*

$$\begin{array}{ll} K_*(A)_{1/n} \simeq K_*(A_l)_{1/n}^G & K_*(V)_{1/n} \simeq K_*(V_l)_{1/n}^G \\ \mathbb{K}_*(A)_{1/n} \simeq \mathbb{K}_*(A_l)_{1/n}^G & \mathbb{K}_*(V)_{1/n} \simeq \mathbb{K}_*(V_l)_{1/n}^G \\ KH_*(A)_{1/n} \simeq KH_*(A_l)_{1/n}^G & KH_*(V)_{1/n} \simeq KH_*(V_l)_{1/n}^G \\ THH_*(A)_{1/n} \simeq THH_*(A_l)_{1/n}^G & THH_*(V)_{1/n} \simeq THH_*(V_l)_{1/n}^G \\ TC_*(A)_{1/n} \simeq TC_*(A_l)_{1/n}^G & TC_*(V)_{1/n} \simeq TC_*(V_l)_{1/n}^G, \end{array}$$

where $(-)_{1/n} := (-) \otimes_{\mathbb{Z}} \mathbb{Z}[1/n]$, $A_l := A \otimes_k l$ and $V_l := V \times_{\text{Spec}(k)} \text{Spec}(l)$. When the characteristic of k does not divide n , one has moreover the canonical isomorphisms

$$\begin{array}{ll} HH_*(A) \simeq HH_*(A_l)^G & HH_*(V) \simeq HH_*(V_l)^G \\ HC_*(A) \simeq HC_*(A_l)^G & HC_*(V) \simeq HC_*(V_l)^G \\ HP_*(A) \simeq HP_*(A_l)^G & HP_*(V) \simeq HP_*(V_l)^G \\ HN_*(A) \simeq HN_*(A_l)^G & HN_*(V) \simeq HN_*(V_l)^G. \end{array}$$

Remark 1.4. The above isomorphisms concerning algebraic K -theory, topological Hochschild homology, and topological cyclic homology hold more intrinsically in the category of spectra localized at the $\mathbb{Z}[1/n]$ -linear stable equivalences. On the other hand, the above isomorphisms concerning cyclic homology (and all its variants) hold more intrinsically in the derived category of mixed complexes; see [3, §5.3].

The isomorphism $HH_*(A) \simeq HH_*(A_l)^G$ was established by Geller-Weibel [2] in the particular case of a *commutative* k -algebra A . Under the assumption that k is of characteristic zero, the isomorphism $HC_*(A) \simeq HC_*(A_l)^G$ was also proved by the authors. In what concerns algebraic K -theory, the isomorphism $K_*(V)_{1/n} \simeq K_*(V_l)_{1/n}^G$ is (well) known to the experts². Besides these examples in the “commutative world”, all the remaining isomorphisms of Corollary 1.3 (and of Theorem 1.2) are, to the best of the author’s knowledge, new in the literature. Due to its generality and simplicity, we believe that Theorem 1.2 will soon be part of the toolkit of every mathematician whose research comes across the above conditions (i)–(ii).

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²I believe the credit should be given to Thomason [16].

2. BACKGROUND ON DG CATEGORIES

Let $\mathcal{C}(k)$ be the category of cochain complexes of k -vector spaces; we use cohomological notation. A *differential graded (=dg) category* \mathcal{A} is a category enriched over $\mathcal{C}(k)$ (morphisms sets $\mathcal{A}(x, y)$ are complexes) in such a way that composition fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} f \circ d(g)$. A *dg functor* $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor enriched over $\mathcal{C}(k)$; consult Keller's ICM survey [3] for further details.

Products. The *cartesian product* $\mathcal{A} \times \mathcal{B}$ (resp. *tensor product* $\mathcal{A} \otimes_k \mathcal{B}$) of two dg categories \mathcal{A} and \mathcal{B} is defined as follows: the set of objects is the cartesian product of the sets of objects of \mathcal{A} and \mathcal{B} and the complexes of morphisms are given by $(\mathcal{A} \times \mathcal{B})((x, z), (y, w)) := \mathcal{A}(x, y) \times \mathcal{B}(z, w)$ (resp. by $(\mathcal{A} \otimes_k \mathcal{B})((x, z), (y, w)) := \mathcal{A}(x, y) \otimes_k \mathcal{B}(z, w)$). As explained in [3, §2.3], the tensor product gives rise to a symmetric monoidal structure on $\mathbf{dgc}at$ with \otimes -unit the dg category \underline{k} .

Modules. Let \mathcal{A} be a dg category. Its *opposite* dg category \mathcal{A}^{op} has the same objects and complexes of morphisms given by $\mathcal{A}^{\text{op}}(x, y) := \mathcal{A}(y, x)$. A *left* (resp. *right*) \mathcal{A} -*module* M is a dg functor $M : \mathcal{A} \rightarrow \mathcal{C}(k)_{\text{dg}}$ (resp. $M : \mathcal{A}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$) with values in the dg category $\mathcal{C}_{\text{dg}}(k)$ of cochain complexes of k -vector spaces. Let us denote by $\mathcal{C}(\mathcal{A})$ the category of right \mathcal{A} -modules; see [3, §2.3]. Recall from [3, §3.2] that the *derived category* $\mathcal{D}(\mathcal{A})$ of \mathcal{A} is the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of objectwise quasi-isomorphisms. Its full subcategory compact objects (see [9, Def. 4.2.7]) will be denoted by $\mathcal{D}_c(\mathcal{A})$.

Morita equivalences. A dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is called a *Morita equivalence* if the restriction of scalars functor $\mathcal{D}(\mathcal{B}) \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$ is an equivalence of (triangulated) categories; see [3, §4.6].

Bimodules. Let \mathcal{A} and \mathcal{B} be two dg categories. A \mathcal{A} - \mathcal{B} -*bimodule* X is a dg functor $X : \mathcal{A} \otimes_k \mathcal{B}^{\text{op}} \rightarrow \mathcal{C}_{\text{dg}}(k)$, i.e. a right $(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})$ -module. A standard example is given by the \mathcal{A} - \mathcal{A} -bimodule

$$(2.1) \quad \mathcal{A}(-, -) : \mathcal{A} \otimes_k \mathcal{A}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k) \quad (x, y) \mapsto \mathcal{A}(y, x).$$

Let us denote by $\text{rep}(\mathcal{A}, \mathcal{B})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A}^{\text{op}} \otimes_k \mathcal{B})$ consisting of those \mathcal{A} - \mathcal{B} -bimodules X such that for every object x of \mathcal{A} the right \mathcal{B} -module $X(x, -)$ belongs to $\mathcal{D}_c(\mathcal{B})$. Associated to a dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$ we have the \mathcal{A} - \mathcal{B} -bimodule

$$X_F : \mathcal{A} \otimes_k \mathcal{B}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k) \quad (x, z) \mapsto \mathcal{B}(z, F(x))$$

as well as the \mathcal{B} - \mathcal{A} -bimodule

$${}_F X : \mathcal{B} \otimes_k \mathcal{A}^{\text{op}} \longrightarrow \mathcal{C}_{\text{dg}}(k) \quad (z, x) \mapsto \mathcal{B}(F(x), z).$$

Clearly, X_F belongs to $\text{rep}(\mathcal{A}, \mathcal{B})$. In contrast, ${}_F X$ belongs to $\text{rep}(\mathcal{B}, \mathcal{A})$ if and only if for every $z \in \mathcal{B}$ the right \mathcal{A} -module $x \mapsto \mathcal{B}(F(x), z)$ belongs to $\mathcal{D}_c(\mathcal{A})$.

Remark 2.2. Given a dg category \mathcal{D} and a dg functor $F : \mathcal{A} \rightarrow \mathcal{B}$, we have an equality $(\text{id} \otimes F)X = \mathcal{D}(-, -) \otimes_k {}_F X$ of $(\mathcal{D} \otimes_k \mathcal{B})$ -($\mathcal{D} \otimes_k \mathcal{A}$)-bimodules, where $\text{id} \otimes F$ stands for the dg functor obtained by tensoring \mathcal{D} with F .

3. NONCOMMUTATIVE MOTIVES

In this section we recall from [14] the construction of the additive category \mathbf{Hmo}_0 of noncommutative motives; consult also the survey article [11]. This category, as well as its universal property, will play a key role in the proof of Theorem 1.2.

Consider first the category \mathbf{Hmo} with the same objects as $\mathbf{dgc}at$, with morphisms defined as $\mathrm{Hom}_{\mathbf{Hmo}}(\mathcal{A}, \mathcal{B}) := \mathrm{Iso} \, \mathrm{rep}(\mathcal{A}, \mathcal{B})$ where Iso denotes the set of isomorphism classes, and with composition law given by the tensor product of bimodules

$$\mathrm{Iso} \, \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times \mathrm{Iso} \, \mathrm{rep}(\mathcal{B}, \mathcal{C}) \longrightarrow \mathrm{Iso} \, \mathrm{rep}(\mathcal{A}, \mathcal{C}) \quad (X, Y) \mapsto X \otimes_{\mathcal{B}} Y.$$

Note that the identity of every $\mathcal{A} \in \mathbf{Hmo}$ is given by the above \mathcal{A} - \mathcal{A} -bimodule (2.1) and that we have a natural functor

$$(3.1) \quad \mathbf{dgc}at \longrightarrow \mathbf{Hmo} \quad (F : \mathcal{A} \rightarrow \mathcal{B}) \mapsto X_F.$$

As explained in *loc. cit.*, the symmetric monoidal structure on $\mathbf{dgc}at$ descends to \mathbf{Hmo} making (3.1) into a symmetric monoidal functor.

The *additivization* of \mathbf{Hmo} is the additive category \mathbf{Hmo}_0 with the same objects as \mathbf{Hmo} , with abelian groups of morphisms given by $\mathrm{Hom}_{\mathbf{Hmo}_0}(\mathcal{A}, \mathcal{B}) := K_0 \, \mathrm{rep}(\mathcal{A}, \mathcal{B})$ where K_0 stands for the Grothendieck group, and with composition law induced by the tensor product of bimodules

$$K_0 \, \mathrm{rep}(\mathcal{A}, \mathcal{B}) \times K_0 \, \mathrm{rep}(\mathcal{B}, \mathcal{C}) \longrightarrow K_0 \, \mathrm{rep}(\mathcal{A}, \mathcal{C}) \quad ([X], [Y]) \mapsto [X \otimes_{\mathcal{B}} Y].$$

Note that the products (=direct sums) in \mathbf{Hmo}_0 are given by the cartesian product of dg categories. Note that we have also a natural functor

$$(3.2) \quad \mathbf{Hmo} \longrightarrow \mathbf{Hmo}_0 \quad X \mapsto [X].$$

As explained in *loc. cit.*, the symmetric monoidal structure on \mathbf{Hmo} descends furthermore to a bilinear symmetric monoidal structure on \mathbf{Hmo}_0 making (3.2) into a symmetric monoidal functor. As proved in [14, Thms. 5.3 and 6.3], the composition

$$U : \mathbf{dgc}at \xrightarrow{(3.1)} \mathbf{Hmo} \xrightarrow{(3.2)} \mathbf{Hmo}_0$$

is the *universal additive invariant*, i.e. given any additive category \mathbf{D} there is an induced equivalence of categories

$$(3.3) \quad U^* : \mathrm{Fun}_{\mathrm{add}}(\mathbf{Hmo}_0, \mathbf{D}) \xrightarrow{\sim} \mathrm{Fun}_{\mathbf{A}}(\mathbf{dgc}at, \mathbf{D}),$$

where the left-hand-side denotes the category of additive functors and the right-hand-side the category of additive invariants in the sense of Definition 1.1. Because of this universal property, which is reminiscent from motives, \mathbf{Hmo}_0 is called the (additive) category of *noncommutative motives*.

4. PROOF OF THEOREM 1.2

Recall first from Galois theory that the order of the Galois group G is n . Let us denote by $\iota : k \hookrightarrow l$ the finite Galois field extension l/k . Note that by definition \mathcal{A}_l identifies with the dg category $\mathcal{A} \otimes_k l$ (where l is considered as a k -algebra) and $\iota_{\mathcal{A}}$ identifies with the dg functor $\mathrm{id} \otimes_l : \mathcal{A} \simeq \mathcal{A} \otimes_k k \rightarrow \mathcal{A} \otimes_k l$.

Lemma 4.1. *The $(\mathcal{A} \otimes_k l)$ - \mathcal{A} -bimodule $\iota_{\mathcal{A}} X$ belongs to $\mathrm{rep}(\mathcal{A} \otimes_k l, \mathcal{A})$. Consequently, it gives rise to a well-defined morphism $[\iota_{\mathcal{A}} X] : U(\mathcal{A} \otimes_k l) \rightarrow U(\mathcal{A})$ in \mathbf{Hmo}_0 .*

Proof. By applying Remark 2.2 to $\mathcal{D} := \mathcal{A}$ and $F := \underline{L}$ we obtain the following equalities of $(\mathcal{A} \otimes_k \underline{L})$ - $(\mathcal{A} \otimes_k \underline{k})$ -bimodules

$$(4.2) \quad {}_{\iota_{\mathcal{A}}}X = (\text{id} \otimes \underline{L})X = \mathcal{A}(-, -) \otimes_k \underline{L}X.$$

Since the \mathcal{A} - \mathcal{A} -bimodule $\mathcal{A}(-, -)$ is the identity of $\mathcal{A} \in \mathbf{Hmo}$ and the category \mathbf{Hmo} is symmetric monoidal, it suffices then to show that the \underline{L} - \underline{k} -bimodule $\underline{L}X$ belongs to $\text{rep}(\underline{L}, \underline{k})$. Note that $\underline{L}X$ is simply l (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the left multiplication by l and with the right multiplication by k . Since by hypothesis the field extension l/k is finite, $\underline{L}X$ is a compact object in $\mathcal{D}(\underline{k})$. Consequently, the \underline{L} - \underline{k} -bimodule $\underline{L}X$ belongs to $\text{rep}(\underline{L}, \underline{k})$ and so the proof is finished. \square

Proposition 4.3. *The morphisms $[\iota_{\mathcal{A}}X] : U(\mathcal{A} \otimes_k \underline{L}) \rightarrow U(\mathcal{A})$ and $U(\iota_{\mathcal{A}}) : U(\mathcal{A}) \rightarrow U(\mathcal{A} \otimes_k \underline{L})$ in \mathbf{Hmo}_0 verify the equality $[\iota_{\mathcal{A}}X] \circ U(\iota_{\mathcal{A}}) = n \cdot \text{id}_{U(\mathcal{A})}$.*

Proof. Recall that the functor $\mathbf{Hmo} \rightarrow \mathbf{Hmo}_0$ is symmetric monoidal and that $[\mathcal{A}(-, -)] \in K_0 \text{rep}(\mathcal{A}, \mathcal{A})$ is the identity $\text{id}_{U(\mathcal{A})}$ of $U(\mathcal{A}) \in \mathbf{Hmo}_0$. By combining these facts with the above equality (4.2) of $(\mathcal{A} \otimes_k \underline{L})$ - $(\mathcal{A} \otimes_k \underline{k})$ -bimodules, we observe that $[\iota_{\mathcal{A}}X]$ identifies with

$$(4.4) \quad \text{id}_{U(\mathcal{A})} \otimes [\underline{L}X] : U(\mathcal{A}) \otimes_k U(\underline{L}) \longrightarrow U(\mathcal{A}) \otimes_k U(\underline{k}).$$

Since the functor U is symmetric monoidal, $U(\iota_{\mathcal{A}})$ admits also the following description

$$(4.5) \quad \text{id}_{U(\mathcal{A})} \otimes U(\underline{L}) : U(\mathcal{A}) \otimes_k U(\underline{k}) \longrightarrow U(\mathcal{A}) \otimes_k U(\underline{L}).$$

By combining (4.4)-(4.5) with the fact that the symmetric monoidal structure on \mathbf{Hmo}_0 is bilinear, we conclude then that it suffices to verify the equality $[\underline{L}X] \circ U(\underline{L}) = n \cdot \text{id}_{U(\underline{k})}$. As explained in §3, the composition

$$U(\underline{k}) \xrightarrow{U(\underline{L})} U(\underline{L}) \xrightarrow{[\underline{L}X]} U(\underline{k})$$

is given by $[X_{\underline{L}} \otimes_{\underline{L}} \underline{L}X] \in K_0 \text{rep}(\underline{k}, \underline{k})$. Note that the \underline{k} - \underline{k} -bimodule $X_{\underline{L}} \otimes_{\underline{L}} \underline{L}X$ is simply $l \otimes_l l \simeq l$ (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the left and right multiplication by k . Recall that by hypothesis the field extension l/k is finite of degree n . Therefore, by choosing a k -linear basis $\{b_1, \dots, b_n\}$ of l , we obtain an isomorphism of \underline{k} - \underline{k} -bimodules

$$(4.6) \quad X_{\underline{L}} \otimes_{\underline{L}} \underline{L}X \simeq \underline{k}(-, -)^{\oplus n}.$$

Now, recall that the identity $\text{id}_{U(\underline{k})}$ of $U(\underline{k}) \in \mathbf{Hmo}_0$ is the class $[\underline{k}(-, -)] \in K_0 \text{rep}(\underline{k}, \underline{k})$ of the \underline{k} - \underline{k} -bimodule $\underline{k}(-, -)$. By construction of the Grothendieck group we have $[\underline{k}(-, -)^{\oplus n}] = n \cdot [\underline{k}(-, -)]$ in $K_0 \text{rep}(\underline{k}, \underline{k})$. Therefore, using the above isomorphism (4.6), we obtain the searched equality

$$[\underline{L}X] \circ U(\underline{L}) = [X_{\underline{L}} \otimes_{\underline{L}} \underline{L}X] = [\underline{k}(-, -)^{\oplus n}] = n \cdot [\underline{k}(-, -)] = n \cdot \text{id}_{U(\underline{k})}.$$

\square

Given an element $\sigma \in G$, let $\sigma_{\mathcal{A}}$ be the dg functor $\text{id} \otimes \underline{\sigma} : \mathcal{A} \otimes_k \underline{L} \xrightarrow{\sim} \mathcal{A} \otimes_k \underline{L}$.

Proposition 4.7. *The morphisms $[\iota_{\mathcal{A}}X] : U(\mathcal{A} \otimes_k \underline{L}) \rightarrow U(\mathcal{A})$ and $U(\iota_{\mathcal{A}}) : U(\mathcal{A}) \rightarrow U(\mathcal{A} \otimes_k \underline{L})$ in \mathbf{Hmo}_0 verify the equality $U(\iota_{\mathcal{A}}) \circ [\iota_{\mathcal{A}}X] = \sum_{\sigma \in G} U(\sigma_{\mathcal{A}})$.*

Proof. Since the functor U is symmetric monoidal and the symmetric monoidal structure on \mathbf{Hmo}_0 is bilinear, the morphism $\sum_{\sigma \in G} U(\sigma_{\mathcal{A}})$ identifies with

$$(4.8) \quad \text{id} \otimes \left(\sum_{\sigma \in G} U(\underline{\sigma}) \right) : U(\mathcal{A}) \otimes_k U(\underline{l}) \longrightarrow U(\mathcal{A}) \otimes_k U(\underline{l}).$$

By combining (4.8) with the above descriptions (4.4)-(4.5) of $[\iota_{\mathcal{A}} X]$ and $U(\iota_{\mathcal{A}})$, we conclude then that it suffices to verify the equality $U(\underline{l}) \circ [\underline{l} X] = \sum_{\sigma \in G} U(\underline{\sigma})$.

Recall from Thomason [16, Example 1.50] that, since by hypothesis the finite field extension l/k is Galois, we have the following isomorphism of k -algebras

$$\theta : l \otimes_k l \xrightarrow{\sim} \prod_{\sigma \in G} l \quad b_1 \otimes b_2 \mapsto (b_1 b_2, \dots, b_1 \sigma(b_2), \dots).$$

Let us denote by ϵ_1 (resp. by ϵ_2) the k -algebra homomorphism $l \rightarrow l \otimes_k l$ that maps b to $b \otimes 1$ (resp. to $1 \otimes b$), and write η_1 (resp. η_2) for the composition $\theta \circ \epsilon_1$ (resp. $\theta \circ \epsilon_2$). Since $\underline{l} \otimes_k \underline{l} = \underline{l} \otimes_k \underline{l}$ and $\prod_{\sigma \in G} \underline{l} = \prod_{\sigma \in G} \underline{l}$, we have then the following commutative diagram in $\mathbf{dgc}at$:

$$(4.9) \quad \begin{array}{ccccc} \underline{k} & \xrightarrow{\underline{l}} & \underline{l} & \xlongequal{\quad} & \underline{l} \\ \downarrow \underline{l} & & \downarrow \epsilon_1 & & \downarrow \eta_1 \\ \underline{l} & \xrightarrow{\epsilon_2} & \underline{l} \otimes_k \underline{l} & \xrightarrow[\sim]{\theta} & \prod_{\sigma \in G} \underline{l} \\ & \searrow \eta_2 & & & \end{array}$$

Note that $\eta_1 = \{\text{id}_{\underline{l}}\}_{\sigma \in G} : \underline{l} \rightarrow \prod_{\sigma \in G} \underline{l}$. As a consequence, the right \underline{l} -module $\eta_1 X$ is simply $\prod_{\sigma \in G} \underline{l} = \underline{l}^{\oplus n}$ (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the *diagonal* right multiplication by \underline{l} . This is a compact object in $\mathcal{D}(\underline{l})$ and so the $(\prod_{\sigma \in G} \underline{l})$ - \underline{l} -bimodule $\eta_1 X$ belongs to $\text{rep}(\prod_{\sigma \in G} \underline{l}, \underline{l})$. As explained in §3, we obtain then a well-defined morphism $[\eta_1 X] : U(\prod_{\sigma \in G} \underline{l}) \rightarrow U(\underline{l})$ in \mathbf{Hmo}_0 . Let us now show that the following two compositions agree

$$(4.10) \quad U(\underline{l}) \xrightarrow{[\underline{l} X]} U(\underline{k}) \xrightarrow{U(\underline{l})} U(\underline{l}) \quad U(\underline{l}) \xrightarrow{U(\eta_2)} U\left(\prod_{\sigma \in G} \underline{l}\right) \xrightarrow{[\eta_1 X]} U(\underline{l}).$$

The composition of the left-hand-side is given by the class $[\underline{l} X \otimes_k X_{\underline{l}}] \in K_0 \text{rep}(\underline{l}, \underline{l})$ of the \underline{l} - \underline{l} -bimodule $\underline{l} X \otimes_k X_{\underline{l}}$, which is $l \otimes_k l$ (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the left and right multiplication by \underline{l} . On the other hand, the composition of the right-hand-side is given by the class $[X_{\eta_2} \otimes_{(\prod_{\sigma \in G} \underline{l})} \eta_1 X] \in K_0 \text{rep}(\underline{l}, \underline{l})$. Using θ and the commutativity of the above diagram (4.9), we obtain an isomorphism of \underline{l} - \underline{l} -bimodules

$$(4.11) \quad X_{\eta_2} \otimes_{(\prod_{\sigma \in G} \underline{l})} \eta_1 X \simeq X_{\epsilon_2} \otimes_{(l \otimes_k l)} \epsilon_1 X.$$

Note that the right-hand-side of (4.11) is simply $(l \otimes_k l) \otimes_{(l \otimes_k l)} (l \otimes_k l)$ (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the left and right multiplications by \underline{l} . Therefore, the canonical isomorphism of l - l -bimodules

$$(l \otimes_k l) \otimes_{(l \otimes_k l)} (l \otimes_k l) \simeq l \otimes_k l$$

allows us to conclude that $[\underline{l} X \otimes_k X_{\underline{l}}]$ equals $[X_{\eta_2} \otimes_{(\prod_{\sigma \in G} \underline{l})} \eta_1 X]$. As a consequence, the above compositions (4.10) agree and so we obtain the equality

$$(4.12) \quad U(\underline{l}) \circ [\underline{l} X] = [\eta_1 X] \circ [U(\eta_2)].$$

Now, note that $\underline{\eta}_2 = \{\underline{\sigma}\}_{\sigma \in G} : \underline{L} \rightarrow \prod_{\sigma \in G} \underline{L}$. Since the functor U preserves (finite) products the composition of the right-hand-side of (4.10) identifies with

$$U(\underline{L}) \xrightarrow{\{U(\underline{\sigma})\}_{\sigma \in G}} \prod_{\sigma \in G} U(\underline{L}) \xrightarrow{[\underline{\eta}_1 X]} U(\underline{L}).$$

By applying Lemma 4.14 to the morphism $\{U(\underline{\sigma})\}_{\sigma \in G}$ we conclude then that

$$(4.13) \quad [\underline{\eta}_1 X] \circ [U(\underline{\eta}_2)] = [\underline{\eta}_1 X] \circ \{U(\underline{\sigma})\}_{\sigma \in G} = \sum_{\sigma \in G} U(\underline{\sigma}).$$

Finally, by combining (4.12)-(4.13) we obtain the searched equality $U(\underline{L}) \circ [\underline{L} X] = \sum_{\sigma \in G} U(\underline{\sigma})$. \square

Lemma 4.14. *For every morphism $\{[Y_\sigma]\}_{\sigma \in G}$ in \mathbf{Hmo}_0 from $U(\underline{L})$ to $\prod_{\sigma \in G} U(\underline{L})$, we have the following equality $[\underline{\eta}_1 X] \circ \{[Y_\sigma]\}_{\sigma \in G} = \sum_{\sigma \in G} [Y_\sigma]$.*

Proof. As explained in §3, the products in \mathbf{Hmo}_0 are given by the cartesian product of dg categories. Hence, the composition $[\underline{\eta}_1 X] \circ \{[Y_\sigma]\}_{\sigma \in G}$ identifies with

$$[(\prod_{\sigma \in G} Y_\sigma) \otimes_{(\prod_{\sigma \in G} \underline{L})} \underline{\eta}_1 X] \in K_0 \text{rep}(\underline{L}, \underline{L}).$$

Since $\underline{\eta}_1 = \{\text{id}_{\underline{L}}\}_{\sigma \in G} : \underline{L} \rightarrow \prod_{\sigma \in G} \underline{L}$, the $(\prod_{\sigma \in G} \underline{L})$ - \underline{L} -bimodule $\underline{\eta}_1 X$ is simply $\prod_{\sigma \in G} l$ (considered as a complex of k -vector spaces concentrated in degree zero) endowed with the left multiplication by $\prod_{\sigma \in G} l$ and with the *diagonal* right multiplication by l . As a consequence, we have the following isomorphism of \underline{L} - \underline{L} -bimodules

$$(\prod_{\sigma \in G} Y_\sigma) \otimes_{(\prod_{\sigma \in G} \underline{L})} \underline{\eta}_1 X \simeq \prod_{\sigma \in G} Y_\sigma.$$

The proof follows now from the standard equalities $[\prod_{\sigma \in G} Y_\sigma] = [\bigoplus_{\sigma \in G} Y_\sigma] = \sum_{\sigma \in G} [Y_\sigma]$ in $K_0 \text{rep}(\underline{L}, \underline{L})$. \square

We now have all the ingredients needed for the conclusion of the proof of Theorem 1.2. By hypothesis, $E : \mathbf{dgc} \rightarrow \mathbf{D}$ is an additive invariant. Hence, thanks to equivalence (3.3), there exists an additive functor \overline{E} making the following diagram commute

$$(4.15) \quad \begin{array}{ccc} \mathbf{dgc} & \xrightarrow{E} & \mathbf{D} \\ U \downarrow & \nearrow \overline{E} & \\ \mathbf{Hmo}_0 & & \end{array}.$$

Using Propositions 4.3 and 4.7, we obtain then well-defined morphisms

$$E(\iota_{\mathcal{A}}) : E(\mathcal{A}) \longrightarrow E(\mathcal{A} \otimes_k \underline{L}) \quad \overline{E}([\iota_{\mathcal{A}} X]) : E(\mathcal{A} \otimes_k \underline{L}) \longrightarrow E(\mathcal{A})$$

verifying the equalities

$$(4.16) \quad \overline{E}([\iota_{\mathcal{A}} X]) \circ E(\iota_{\mathcal{A}}) = n \cdot \text{id}_{E(\mathcal{A})} \quad E(\iota_{\mathcal{A}}) \circ \overline{E}([\iota_{\mathcal{A}} X]) = \sum_{\sigma \in G} E(\sigma_{\mathcal{A}}).$$

Since by hypothesis the additive category \mathbf{D} is idempotent complete and $\mathbb{Z}[1/n]$ -linear, the G -invariant part $E(\mathcal{A} \otimes_k \underline{L})^G$ of $E(\mathcal{A} \otimes_k \underline{L})$ exists and is given by the image of the idempotent endomorphism $1/n \sum_{\sigma \in G} E(\sigma_{\mathcal{A}})$ of $E(\mathcal{A} \otimes_k \underline{L})$. We obtain

then a well-defined inclusion morphism $\text{inc} : E(\mathcal{A} \otimes_k \mathbb{L})^G \hookrightarrow E(\mathcal{A} \otimes_k \mathbb{L})$ such that $E(\sigma_{\mathcal{A}}) \circ \text{inc} = \text{inc}$ for every $\sigma \in G$. Note that the following equalities

$$\left(\frac{1}{n} \sum_{\sigma \in G} E(\sigma_{\mathcal{A}}) \right) \circ E(\iota_{\mathcal{A}}) = \frac{1}{n} \sum_{\sigma \in G} E(\sigma_{\mathcal{A}} \circ \iota_{\mathcal{A}}) = \frac{1}{n} \sum_{\sigma \in G} E(\iota_{\mathcal{A}}) = E(\iota_{\mathcal{A}})$$

imply that $E(\iota_{\mathcal{A}})$ factors through inc . Let us now show that the morphisms

$$(4.17) \quad E(\mathcal{A}) \xrightarrow{E(\iota_{\mathcal{A}})} E(\mathcal{A} \otimes_k \mathbb{L})^G \quad E(\mathcal{A} \otimes_k \mathbb{L})^G \xrightarrow{\text{inc}} E(\mathcal{A} \otimes_k \mathbb{L}) \xrightarrow{\overline{E}([\iota_{\mathcal{A}} X])} E(\mathcal{A})$$

are inverse of each other. Clearly, using the left-hand-side of (4.16) we obtain

$$(4.18) \quad \overline{E}([\iota_{\mathcal{A}} X]) \circ \text{inc} \circ E(\iota_{\mathcal{A}}) = n \cdot \text{id}_{E(\mathcal{A})} .$$

On the other hand, by combining the equality $E(\sigma_{\mathcal{A}}) \circ \text{inc} = \text{inc}$ with the right-hand-side of (4.16), we obtain

$$(4.19) \quad E(\iota_{\mathcal{A}}) \circ \overline{E}([\iota_{\mathcal{A}} X]) \circ \text{inc} = n \cdot \text{id}_{E(\mathcal{A} \otimes_k \mathbb{L})} .$$

Since by hypothesis D is $\mathbb{Z}[1/n]$ -linear, both compositions (4.18)-(4.19) are isomorphisms. This implies that the above morphisms (4.17) are inverse of each other and so the proof is finished.

5. PROOF OF COROLLARY 1.3

Recall from Loday [7, §1-5] and from Keller [3, §5.3] that by construction the abelian groups HH_* , HC_* , HP_* and HN_* are in fact k -vector spaces. Therefore, when the characteristic of k does *not* divide n these groups are also $\mathbb{Z}[1/n]$ -modules. In what concerns (quasi-compact separated) k -schemes V , the isomorphisms of Corollary 1.3 follow from the following Morita equivalences

$$(5.1) \quad \begin{aligned} \mathcal{D}_{\text{perf}}^{\text{dg}}(V) \otimes_k \mathbb{L} &\simeq \mathcal{D}_{\text{perf}}^{\text{dg}}(V) \otimes_k \mathcal{D}_{\text{perf}}^{\text{dg}}(\text{Spec}(l)) \\ &\simeq \mathcal{D}_{\text{perf}}^{\text{dg}}(V \times_{\text{Spec}(k)} \text{Spec}(l)) \\ &= \mathcal{D}_{\text{perf}}^{\text{dg}}(V_l) , \end{aligned}$$

where (5.1) is a particular case of [15, Prop. 6.2].

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